Method of Orthogonal Loadings—A Fresh Look

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Introduction

GRAHAM'S paper[1] on the use of orthogonal loadings as a method of obtaining minimum drag wing shapes seems to have largely gone unnoticed. The reason appears to lie in the tedious method he had proposed to obtain a set of orthogonal loadings that was not suitable for digital computers. In this Note we show how Graham's method could be adopted for automatic calculations.

The drag, in orthogonal loadings, has the great merit of sharing with the velocity components, the pressure and the downwash distribution, the property of superposition. It was further shown that if one obtains a set of lifting pressure loadings, any linear combination of its members, each carrying positive lift, would give smaller drag for a given total lift than any single member carrying the same total lift. It may also be shown that the absolute minimum may be reached as close as desired by adding additional loadings to the set. This immediately shows that orthogonal loadings may be used in a constructive way to design minimum drag wings. The minimum drag theorems of Munk[2] and Jones[3] have not been very useful in this respect since they only tell us the properties of minimum drag wings in a combined flowfield, and say nothing about the value of the minimum drag, or how to achieve it. Graham's method, to a large extent, overcomes these difficulties. The results are valid when leading-edge singularities are not present.

Graham's Method of Orthogonalization

Let $P_i \equiv (p_i, \alpha_i)$ be a pressure loading, where p_i is the wing surface pressure coefficient distribution due to an incidence distribution α_i. Two loadings P_i, P_j are of different types if

$$p_i \neq \epsilon p_j$$

where ϵ is a constant, called the intensity of loading. If we now define the drag coefficient due to two pressure loadings

$$P_i, P_j \text{ as } \int (p_i\alpha_i + p_j\alpha_j)ds + \int (p_i\alpha_j + p_j\alpha_i)ds,$$

then these two loadings are said to be mutually orthogonal if their interference drag is zero; i.e.,

$$\int (p_i\alpha_j + p_j\alpha_i)\, ds = 0 \tag{1}$$

where the integration is over the entire wing surface. It is also assumed that the loadings do not have leading edge singularities.

Now suppose $P = (P_1, P_2, \ldots, P_m)$ is a set of nonorthogonal loadings and $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2 \ldots, \mathbf{P}_m)$ is an orthogonal set derived from P. The first member of $\mathbf{P}$ is arbitrarily chosen as

$$\mathbf{p}_1 = p_1, \ \boldsymbol{\alpha}_1 = \alpha_1 \tag{2}$$

The second member is taken to be of the form

$$\mathbf{p}_2 = p_1 + cp_2, \ \boldsymbol{\alpha}_2 = \alpha_1 + c\alpha_2 \tag{3}$$

Using the orthogonality criterion

$$\int (\mathbf{p}_2\boldsymbol{\alpha}_1 + \mathbf{p}_1\boldsymbol{\alpha}_2)\, ds = 0$$

gives

$$c = -2d_{11}/(d_{12} + d_{21}) \tag{4}$$

where

$$d_{ij} = \int p_i\alpha_j ds \tag{5}$$

The third member $\mathbf{P}_3$ is obtained in a similar fashion. Let

$$\mathbf{p}_3 = p_1 + c_2p_2 + c_3p_3$$
$$\boldsymbol{\alpha}_3 = \alpha_1 + c_2\alpha_2 + c_3\alpha_3 \tag{6}$$

where c_2 and c_3 are constants. Using the orthogonality criterion, we have

$$\int (\mathbf{p}_3\boldsymbol{\alpha}_1 + \mathbf{p}_1\boldsymbol{\alpha}_3)\, ds = 0$$
$$\int (\mathbf{p}_3\boldsymbol{\alpha}_2 + \mathbf{p}_2\boldsymbol{\alpha}_3)\, ds = 0 \tag{7}$$

from which we deduce

$$2d_{11} + c_2(d_{12} + d_{21}) + c_3(d_{13} + d_{31}) = 0$$
$$(d_{12} + d_{21}) + 2c_2d_{22} + c_3(d_{23} + d_{32}) = 0 \tag{8}$$

which may be solved for c_2 and c_3.

Clearly the process becomes laborious for successive members in the set $\mathbf{P}$ and is not suitable for digital computers. Further, the orthogonal set is not unique since the process could have been started with any member of the set P instead of the first one.

A Rational Approach

We now place the orthogonalization process on a more formal footing and show that it amounts to diagonalizing a certain matrix, a process which can be achieved in any number of ways.[4] This nonuniqueness may be exploited to suit different kinds of computational requirements.

The drag expression may be written as

$$C_D = EDE^T \tag{9}$$

where $E = \{\epsilon_1, \epsilon_2 \ldots, \epsilon_n\}$ is a row vector in load intensities, and D is a symmetric matrix, the elements d_{ij} of which may be defined, without any loss of generality, as

$$d_{ij} = \frac{1}{2} \int (p_i\alpha_j + p_j\alpha_i)ds \tag{10}$$

The definition of orthogonal loadings requires that Eq. (9) be expressed in a form such that D is transformed to a diagonal matrix. If D is nonsingular and real, this is possible. Let G be a square matrix such that GDG^T results in a diagonal matrix. Hence

$$C_D = EG^{-1}GDG^TG^{-T}E^T \tag{11}$$
$$= \mathbf{E}\mathbf{D}\mathbf{E}^T$$

where

$$\mathbf{D} = GDG^T \tag{12}$$

is a diagonal matrix, and $\mathbf{E} = E\,G^{-1}$ is the intensity vector in the transformed space.

From matrix theory we know that D does not have a unique diagonal, hence G is not a unique transformation. One may use this nonuniqueness to diagonalize D to suit different conditions. We mention here only two methods of obtaining G. In the first method, the eigenvalues of D are obtained. The corresponding eigenvectors placed column by column may be taken to constitute G, in which

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case, D contains the eigenvalues in the diagonal. G then is an orthonormal modal matrix of D.

$$G = \begin{bmatrix} g_{11} & g_{21} \cdots g_{n1} \\ g_{12} & g_{22} \cdots g_{n2} \\ \vdots & \\ g_{1n} & g_{2n} \cdots g_{nn} \end{bmatrix} \quad (13)$$

and

$$D = \begin{bmatrix} d_{11} & 0 & \ldots & 0 \\ 0 & d_{22} & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & d_{nn} \end{bmatrix} = GDG^T \quad (14)$$

where the jth column in G is the eigenvector corresponding to the jth eigenvalue $\mathbf{d}_{jj}$. This method is easy to use since computer programs for finding the eigenvalues and eigenvectors of a real symmetric matrix are easily available.[5]

The second method is to look for a matrix G which is triangular; i.e.,

$$G = \begin{bmatrix} g_{11} & g_{12} \cdots g_{1n} \\ 0 & g_{22} \cdots g_{2n} \\ \vdots & \\ 0 & 0 \quad \cdots g_{nn} \end{bmatrix} \quad (15)$$

where

$$g_{ii}^2 = A_i/A_{i-1} \quad (16)$$

and A_{i-k} is the determinant obtained by striking out the last k rows and last k columns from the matrix D. The remaining g_{ij} are obtained from

$$d_{jk} = \Sigma g_{ij} g_{ik}, \quad k = j + 1, \ldots, n \quad (17)$$

the procedure outlined by Graham appears to be equivalent to this latter method of diagonalization.

Two-Constraint Minimization Problem

It may now be expected that the drag minimization calculation would be simplified. Consider for example, the problem of obtaining minimum drag subject to given lift and pitching moment constraints. The problem is

$$\text{Minimize} \quad C_D = \Sigma \epsilon_i^2 d_{ii}$$

$$\text{subject to} \quad \Sigma \epsilon_i l_i = C_L \quad (18)$$

$$\Sigma \epsilon_i m_i = C_M$$

where d_{ii}, l_i, *and* m_i are the drag, lift and moment coefficients, respectively, due to unit intensity of the ith member in an orthognal set. C_L and C_M are prescribed lift and moment coefficients, respectively. The corresponding unconstrained problem is

$$\text{Minimize} \quad \Sigma \epsilon_i^2 d_{ii} + \lambda_1(\Sigma \epsilon_i l_i - C_L) + \lambda_2(\Sigma \epsilon_i m_i - C_M) \quad (19)$$

where λ_1 and λ_2 are the Lagrange multipliers. The necessary conditions that Eq. (19) be a minimum are

$$2\epsilon_i d_{ii} + \lambda_1 l_i + \lambda_2 m_i = 0; \; i = 1, 2, \ldots, n \quad (20)$$

$$\Sigma \epsilon_i l_i = C_L \qquad \Sigma \epsilon_i m_i = C_M \quad (21)$$

Multiplying Eq. (20) by ϵ_i and summing over i and using the two constraint equations (21), we have

$$\lambda_1 = -(\lambda_2 C_M + 2C_D)/C_L \quad (22)$$

Substituting for λ_1 in Eq. (20) we have

$$\epsilon_i = \frac{1}{2d_{ii}} \left[\frac{2C_D l_i}{C_L} - \lambda_2 \left(m_i - \frac{C_M l_i}{C_L} \right) \right] \quad (23)$$

and for C_D we find

$$C_D = \frac{C_M^2 \Sigma \frac{l_j^2}{d_{jj}} - 2C_M C_L \Sigma \frac{l_j m_j}{d_{jj}} + C_L^2 \Sigma \frac{m_j^2}{d_{jj}}}{\Sigma \frac{l_j^2}{d_{jj}} \Sigma \frac{m_j^2}{d_{jj}} - \left(\Sigma \frac{l_j m_j}{d_{jj}} \right)^2} \quad (24)$$

Equation (20) then yields

$$\lambda_2 = \left(2C_M - \frac{2C_D}{C_L} \Sigma \frac{l_j m_j}{d_{jj}} \right) \Big/ \left(\frac{C_M}{C_L} \Sigma \frac{l_j m_j}{d_{jj}} - \Sigma \frac{m_j^2}{d_{jj}} \right) \quad (25)$$

When only the lift constraint is given, setting $\lambda_2 = 0$ gives the results obtained in Ref. 1.

The Langrange multiplier method when applied to nonorthogonal loads may give rise to an ill-conditioned matrix and higher precision may be required to obtain reasonable results. The orthogonal loading method alleviates this problem and should be suitable for digital calculations.

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Frequency Determination from Similarity Considerations

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Nomenclature

$\bar{\alpha}_i, \bar{\alpha}_i', \bar{\alpha}_i''$	= bending frequency coefficients
$\bar{\beta}_i, \bar{\beta}_i', \bar{\beta}_i''$	= torsional frequency coefficients
ρ	= fluid density
ρ_s	= structural density
σ_b	= allowable bending stress
σ_s	= allowable shear stress
μ	= aircraft relative mass
ω	= frequency (rad/sec)
a'	= sonic velocity
$\bar{c}$	= mean aerodynamic chord
c_1', c_2', c_1'', c_2''	= frequency coefficients
g	= gravitational acceleration
k_i	= reduced frequency parameter
l	= semispan length
m	= wing mass per unit length
t	= wing thickness at root
E	= modulus of elasticity

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